Search for the end of a path in the d-dimensional grid and in other graphs

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Abstract

We consider the following search problem. Given a graph G with a vertex s there is an unknown path starting from s. In one query we ask a vertex v and the answer is the set of edges of the path incident to v. We want to determine what is the minimal number of queries needed to find the other endvertex of the path. We consider different variants of this problem and examine their relations. We prove a strong connection between this problem and the theory of graph separators. Finally, we consider the case when the graph G is a grid graph, in which case using the connection with separators, we give asymptotically tight bounds (as a function of the size of the grid n, in case the dimension of the grid d is considered to be fixed) for variants of the above search problem. For this we prove a separator theorem about grid graphs. This search problem is a discrete variant of a problem of Hirsch, Papadimitriou and Vavasis about finding Brouwer fixed points, in particular in the discrete setting our results improve their results.

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1 Introduction

This paper deals with the following search problem. Suppose we are given a simple, undirected graph G with a vertex s. The edges of the simple, directed graph D are a subset of the edges of G (if we consider them without directions). D is unknown to us, however, we know that for every vertex its outdegree in D is at most one and the indegree of s is zero. Our goal is to find the end of the path starting at s with as few queries as possible in the worst case. We denote the end of this path by t. We differentiate several models depending on the query types and the input types.

Setting 1: The indegree of every vertex is one, except for s. In this case D consists of one directed path starting at s, and directed cycles.

Setting 2: D is a path. In this case vertices not on the path have in- and outdegree zero.

We also consider two types of allowed queries. In both cases the input is a vertex v but the answers are different.

Query A: One query is to ask a vertex v. If $v = t$ then the answer is that $v = t$. If $v \neq t$, then the answer is the edge of D going out from v or if there is no such edge (only in case of Setting 2), then the answer is that v is not on the path.

Query B: One query is to ask a vertex v, the answer is the at most two edges of D incident to v (together with their directions). In particular if the answer is only one incoming edge then in both settings we know that $v = t$.

We denote by $h_1^A(G)$ the minimal number of queries of type A needed to find t in Setting 1. Similarly $h_1^B(G), h_2^A(G)$, and $h_2^B(G)$ denotes the minimal number of queries of the appropriate type in the appropriate setting.

In case of Query B, one could consider a non-directed version, where D is non-directed and the answer to the query is the at most two incident edges. Obviously this is a harder problem, hence the lower bounds hold, and one can easily modify our proofs to show the same upper bounds in both settings. Note that the proof of Theorem 3.1 does not work in this case though.

Observation 1.1. *$h_1^A(G) \geq h_1^B(G)$ and $h_2^A(G) \geq h_2^B(G)$.*

As most of our bounds work in case of any types of queries, for simplicity we will only mention A in case of an upper bound and B in case of a lower bound. To state some of our results we need to define separators of graphs. Recall that a set $C \subset V$ of vertices is called a separator if $V \setminus C$ is not connected.

Definition 1.2. *Given a graph $G = (V, E)$, a separator C is called an α-separator if every connected component of $V \setminus C$ has cardinality at most $\alpha|V|$.*

Note that in the literature the notation $f(n)$-separator can also be found, where $f(n)$ is an upper bound on the cardinality of C. In this paper it is more straightforward to fix α and then look for the smallest α-separator. Let $s_\alpha(G)$ be the minimum cardinality of an α-separator in G.

Definition 1.3. *A function f is called subhomogeneous if $f(\alpha x) \leq \alpha f(x)$ for any $0 < \alpha \leq 1$ and $x > 0$.*

Our main results about general graphs are the following.

Theorem 1.4. *Let us suppose we are given a subhomogeneous function f and a graph G such that any subgraph D has an α-separator of size at most $f(|V(D)|)$. Then $h_i^A(G) \leq f(|V|)/(1-\alpha)$ for $i = 1, 2$.*

Theorem 1.5. *For any graph G we have $h_2^B(G) \geq s_{1/2}(G)$.*

Note that a similar lower bound for $h_1^B(G)$ is not plausible in general as there are graphs that have only big separators yet there is only a few valid choices for D, thus for t. For example if G contains a vertex of degree one, different from the source, then this vertex has to be the sink. More specifically, take a big complete graph with one of its vertices s as the source and an additional edge going from s to a new vertex t_0, in every valid D the directed path is the edge from s to t_0, thus t is determined without asking any questions - while all separators are big. However, Setting 1 and 2 are equally interesting in the case of grid graphs.

Now we concentrate on grid graphs. Given a d-dimensional integer grid and an axis-parallel cube $C_d(n)$ with length n with two opposite corners $(0, 0, 0.., 0)$ and $(n-1, n-1, \dots, n-1)$. $G_d(n)$ is the graph on the n^d vertices that are contained in $C_d(n)$ and two vertices are connected if their coordinate vectors differ in exactly one coordinate and differ by one. We identify vertices and their coordinate vectors. Without confusion we denote the set of vertices of the cube also by $C_d(n)$. Let $s = \mathbf{0} = (0, 0, \dots, 0)$.

Our main result about grid graphs is the following.

Theorem 1.6. *$h_1^B(G_d(n))$, $h_2^A(G_d(n))$ and $h_2^B(G_d(n))$ are all $O(n^{d-1})$ and $\Omega(n^{d-1}/d)$.*

We note that when considering grid graphs, one can study the related problem that the path in D is monotone. In this case the needed number of queries reduces dramatically. Indeed, the trivial algorithm which follows the path uses at most dn queries (in all variants of the problem). In two dimensions we could improve slightly this upper bound, yet there is a more significant improvement by Sun [8], who proved that $8/5n$ queries are enough in two dimensions. From below, at least n queries are needed regardless of d (Lemma 6 in [3]). This problem resembles the pyramid-path search problem (but it is not exactly the same), where also a lower bound of n is proved for the two-dimensional case [2].

The motivation of Hirsch, Papadimitriou and Vavasis [3] was to prove worst case lower bounds for finding Brouwer fixed points for algorithms using only function evaluation. They have showed a lower bound that is exponential in the dimension, disproving the conjecture that Scarf's algorithm is polynomial. In our language, they have proved that if the path is monotone from the bottom-left corner (with other vertices isolated), then we need at least $n-2$ questions (Lemma 6 in [3]). Furthermore, they have implicitly proved a lower bound of $\Omega(n^{d-2})$ for the general problem (Theorem 5 in [3]). Our paper is an improvement of their result, although we do not use the continuous setting but rather focus only on the discretification of the problem.

As a tool, we will prove a lower bound on the cardinality of separators of grid graphs, which we think is interesting on its own:

Theorem 1.7. *Any α-separator in the grid graph $G_d(n)$ has cardinality at least $(1-\alpha)n^{d-1}/d$ for $\alpha \geq 1/2$.*

2 Upper bounds

Observation 2.1. *Suppose $C \subset V$ cuts G into parts $Y_1, \ldots, Y_k$. If every vertex of C has been asked, it shows which Y_i contains the end-vertex (or the end-vertex is in C, hence already identified) in all settings.*

Proof. In case of Setting 2, the answers clearly show where the path goes into Y_i and where it leaves it, for any $1 \leq i \leq k$. If the path arrives to a component Y_i more times than leaves it, then it must end somewhere in Y_i. If there is no such component, the path must end in the same component where it started.

The case of Setting 1 can be solved by the same argument, as any cycle which arrives to Y_i leaves it the same amount of times.

$\square$

Proof of Theorem 1.4. Let us choose an α-separator C_1 with $|C_1| \leq f(|V|)$ which cuts G into parts $Y_1, \ldots, Y_k$, and ask all the vertices of C_1. By Observation 2.1 we know which part Y_j contains the end-vertex. Let G_1 be G restricted to Y_j and choose an α-separator C_2 of size at most $f(|Y_j|)$, which cuts G_1 into parts $Z_1, \ldots, Z_l$.

Observe that after asking every vertex of C_2 we cannot use Observation 2.1 for G_1 and C_2, as we do not know the start-vertex. In the proof of Observation 2.1 in one of the cases the fact that we know which one of the parts contains the start-vertex was the only thing that showed where the end-vertex is. However, $C_1 \cup C_2$ is a separator of G, which cuts it into parts $Y_1, \ldots, Y_{j-1}, Y_{j+1}, \ldots, Y_k, Z_1, \ldots, Z_l$. Thus after asking every vertex of $C_1 \cup C_2$ we know which part Z_i contains the end-vertex.

After this we can continue the same way, defining G_2 and asking C_3, defining G_3 and asking C_4 and so on, until either the end-vertex is in some C_i and then it is identified or the set of possible positions of the end-vertex is reduced to be in a part which contains only one vertex. As $|V(G_j)| \leq \alpha|V(G_{j-1})|$ for any j, one can easily see that $|V(G_j)| \leq \alpha^j|V|$. As f is subhomogeneous and that implies monotonicity, $f(C_j) \leq f(|V(G_{j-1})|) \leq f(\alpha^{j-1}|V|) \leq \alpha^{j-1}f(|V|)$. Altogether at most $\sum_{j=1}^{\infty} \alpha^{j-1}f(|V|) \leq f(|V|)/(1-\alpha)$ questions were asked. $\square$

A celebrated theorem of Lipton and Tarjan [5] states that planar graphs have 2/3-separators of size at most $2\sqrt{2}(\sqrt{|V|})$. The current best constant is about 1.97 [1]. Thus we have the following corollary.

Corollary 2.2. *If G is planar, then $h_i^A(G) \leq c\sqrt{|V|}$ for $i = 1, 2$.*

Miller, Teng and Vavasis [7] introduced huge classes of graphs, the so-called overlap graphs for every d and proved that every member G of the class has separator of size $O(|V(G)|^{(d-1)/d})$. They mention that any subset of the d-dimensional infinite grid graph belongs to the class of overlap graphs. As $cx^{(d-1)/d}$ is a subhomogeneus function and $|V(G_d(n))| = n^d$, it already implies $h_i^A(G_d(n) = O(n^{d-1})$. Here we show that the multiplicative constant is less than 3.

Theorem 2.3. $h_i^A(G_d(n) \leq (2 + 1/(2^d - 1))n^{d-1}$ *for $i = 1, 2$.*

Proof. We follow the proof of Theorem 1.4, but the cuts we use are always hyperplanes, which cut the current part into two rectangles. More precisely, let $x_1, x_2, \dots x_d$ be the axis, then C_i will always be a hyperplane which is parallel to x_i modulo d, and cuts G_{i-1} into two parts such that both have size at most $|V(G_{i-1})|/2$. One can easily see that it is possible and $|C_{i+1}| = |C_i|/2$, except if d divides i, in which case $|C_{i+1}| = |C_i|$. This means there are at most $\sum_{i=0}^{\infty} 2^{-i} n^{d-1} + \sum_{j=1}^{\infty} 2^{-jd} n^{d-1} = (2 + 1/(2^d - 1)) n^{d-1}$ questions. □

Probably the earliest result concerning separators of graphs is the following: any tree has a 1/2-separator of cardinality 1 [4]. A constant function is not subhomogeneous, hence we cannot use Theorem 1.4 for trees. However, one can easily see that the same arguments lead to an algorithm of length $\log_2 |V|$. On the other hand it is obvious that in case V is a path, $\log_2 |V|$ queries are needed in Setting 2.

3 Lower bounds

Let us suppose we are given an $r_1 \times r_2 \times r_3 \times \dots \times r_d$ grid graph G. Then let $G^{4,4}$ denote the $4r_1 \times 4r_2 \times r_3 \times \dots \times r_d$ grid graph.

Theorem 3.1. *Let G be a grid graph. Then $h_2^B(G) \leq h_1^B(G^{4,4})$.*

Proof. Suppose we are given a grid graph G and an Algorithm A which finds t in $G^{4,4}$ in Setting 1 using queries of type B. We can naturally identify every vertex of G with a 4×4 grid in $G^{4,4}$, i.e. the vertex $(i_1, \dots i_d)$ corresponds to the axis-parallel 4x4 rectangle (we call it a block) $B(v)$ having 16 vertices, whose two opposite corners are $(4i_1 - 3, 4i_2 - 3, i_3, \dots i_d)$ and $(4i_1, 4i_2, i_3, \dots i_d)$.

Now we consider a directed path P in G. We call a system of a directed path and some directed cycles in $G^{4,4}$ *good*, if it satisfies the properties of Setting 1 and goes through exactly those blocks which correspond to the vertices of P, in the same order.

Now we construct good systems. If a vertex $v \in V(G)$ is not on the path, we cover the corresponding block by a cycle. In case of a vertex $v = (i_1, \dots i_d)$ on the path in G, the directed path arrives to the corresponding block $B(v)$ in a corner $p_1(v)$, and goes straight to a neighboring corner $p_2(v)$, where it leaves. The remaining vertices make a 4×3 rectangle, which can be covered

by 2 cycles. To choose the two corners $p_1(v)$ and $p_2(v)$ (the first and last vertex of the path in the block), we check first the parity of $\sum_{j=3}^d i_j$. In case it is even, one of $(4i_1-3, 4i_2-3, i_3, \ldots i_d)$ and $(4i_1, 4i_2, i_3, \ldots i_d)$ is the first vertex $p_1(v)$ and one of the other two corners is the last vertex $p_2(v)$. In case $\sum_{j=3}^d i_j$ is odd, one of $(4i_1-3, 4i_2-3, i_3, \ldots i_d)$ and $(4i_1, 4i_2, i_3, \ldots i_d)$ is the last vertex and one of the other two corners is the first vertex. Finally, when v is the very last vertex on the path, we define $p_1(v)$ similarly, and cover the remaining vertices by a path starting in $p_1(v)$. Note that these properties do not uniquely determine the system.

Now we are ready to define Algorithm B, which finds the endvertex of the path in G in Setting 2. At every step we call Algorithm A, and then answer such a way that at the end we get a good system. If Algorithm A would ask a vertex v in $G^{4,4}$, Algorithm B asks the corresponding vertex v' in G instead (i.e., the vertex v' with $v \in B(v')$). In case the answer is that v' is not on the path, choose an arbitrary cycle in the corresponding block $B(v)$ and answer according to the edges incident to v.

In case the answer is two edges uv and vw, we define 5 edges on the path and two cycles of lengths six. One edge connects the blocks corresponding to u and v, leaving the last vertex of the path in $B(u)$ and arriving to the first vertex of the path in $B(v)$, i.e. this edge is $p_2(u)p_1(v)$. Similarly we add the edge $p_2(v)p_1(w)$. We also add the three edges which connect $p_1(v)$ and $p_2(v)$. Finally we cover the remaining 12 vertices with two cycles.

We still have to tell which one of the two possible first vertices we use as $p_1(v)$, and similarly for the possible last vertices. In case uv is parallel to one of the first two axis, that determines which corner can be $p_1(v)$ and similarly if vw is parallel to one of the first two axis, that determines which corner can be $p_2(v)$. In case u or w were asked in a previous query, that also determined $p_1(v)$ or $p_2(v)$. Otherwise we choose arbitrarily.

One can easily see that this procedure does not result in a contradiction. And even if Algorithm A would know all the answers in $B(v)$, it does not give more information than what Algorithm B knows after asking v. Algorithm A does not finish before Algorithm B finds the endvertex, thus Algorithm A needs at least as many queries as Algorithm B (on the respective graphs), which finishes the proof. $\square$

Next we prove Theorem 1.7 which claims that any α-separator in the grid graph $G_d(n)$ has cardinality at least $(1-\alpha)n^{d-1}/d$ for $\alpha \geq 1/2$.

Proof of Theorem 1.7. We use induction on d, the statement is trivial for

$d = 1$. Let us denote by C an α-separator which separates the vertex set A and B (and $V = A \cup B \cup C$).

Let us choose an arbitrary axis, and denote by $\mathcal{L}$ all the n^{d-1} parallel lines in the grid which go in that direction. Let $\mathcal{L}' \subset \mathcal{L}$ be the subset of those lines which intersect C. Note that every other element of $\mathcal{L}$ contains vertices only from one of A and B. If $|\mathcal{L}'| \geq (1-\alpha)n^{d-1}/d$, then the proof is done. Hence we can suppose $|\mathcal{L}'| < (1-\alpha)n^{d-1}/d$.

Elements of $\mathcal{L}'$ cover less than $(1-\alpha)n^d/d$ points, hence both A and B contain at least $((1-\alpha)d - (1-\alpha))n^d/d$ vertices, which are not covered by elements of $\mathcal{L}'$. This means there are at least $((1-\alpha)d - (1-\alpha))n^{d-1}/d$ elements of $\mathcal{L}$ which contain only vertices of A, and similarly for B. Now consider a hyperplane in the grid, orthogonal to the direction of the lines of $\mathcal{L}$. Clearly it contains at least $((1-\alpha)d - (1-\alpha))n^{d-1}/d$ elements of A and similarly for B, hence we can apply induction on each of those hyperplanes.

By induction, there are at least $((1-\alpha)d-(1-\alpha))n^{d-1}/d(d-1)$ elements of C in every such hyperplane, which gives altogether at least $n((1-\alpha)d - (1-\alpha))n^{d-1}/d(d-1) = (1-\alpha)n^d/d$ elements. $\square$

We are not aware of any results concerning separators of grid graphs. In the continuous case Magazinov [6] proved a lower bound of $(1-\alpha)n^{(d-1)}/2^d$.

Before showing the lower bound for $h_2^B(G)$, we need a technical lemma.

Lemma 3.2. *Let $1 \leq i \leq l$, $m_i \geq 2$ and $x_{i,1} \leq x_{i,2} \leq \ldots \leq x_{i,m_i}$ positive integers. Let us denote $y_i = x_{i,1} + x_{i,2} + \cdots + x_{i,m_i-1}$ and $y_{l+1} = x_{l,m_l} = 1$. If for every i, $x_{i,j} \leq y_{i+1} + \cdots + y_l + y_{l+1}$, $y_{l+1} = 1$ and z is a nonnegative integer with $z \leq N = y_1 + \cdots + y_l + y_{l+1}$, then there is a subset Z of the series of the $x_{i,j}$'s which sums up to z.*

Proof. Let us consider the following greedy algorithm. We proceed the series in the order $x_{1,1}, x_{1,2}, \ldots, x_{1,m_1-1}, x_{2,1}, x_{2,2}, \ldots, x_{2,m_2-1}, \ldots, x_{l,1}, x_{l,2}, \ldots, x_{l,m_i-1}$, and we add an element $x_{i,j}$ to Z if and only if the sum together with it is at most z. We show that this algorithm gives the desired subset at the end.

For that we show that after $x_{1,1}, x_{1,2}, \ldots, x_{1,m_1-1}, x_{2,1}, x_{2,2}, \ldots, x_{2,m_2-1}, \ldots, x_{i,1}, x_{i,2}, \ldots, x_{i,m_i-1}$ the sum is in the interval $[z - (y_{i+1} + \cdots + y_{l+1}), z]$ (this is enough, as at the end this interval has length 0).

We use induction on i. In case $i = 1$, either we can add every $x_{i,j}$ to Z, or not. If we can, the sum is in the interval as it is bigger than $y_1 = N - (y_2 + \cdots + y_{l+1}) \geq z - (y_{i+1} + \cdots + y_{l+1})$. If we cannot, it means

that a certain $x_{1,j}$ would push it over z. But then without that $x_{1,j}$ the sum was greater than $z - x_{1,j} \geq z - (y_2 + \cdots + y_{l+1})$.

The induction step goes similarly. Suppose that the sum after $x_{1,1}, x_{1,2}, \ldots, x_{1,m_1-1}, x_{2,1}, x_{2,2}, \ldots, x_{2,m_2-1}, \ldots, x_{i-1,1}, x_{i-1,2}, \ldots, x_{i-1,m_{i-1}-1}$ is in the interval $[z - (y_i + \cdots + y_{l+1}), z]$, and consider $x_{i,1}, x_{i,2}, \ldots, x_{i,m_i-1}$. If we can add all of them to Z, then the sum increases by y_i, hence it is at least $z - (y_i + \cdots + y_{l+1}) + y_i$. If not, then again a certain $x_{i,j}$ would push it over z, which means the sum was already greater than $z - x_{i,j} \geq z - (y_{i+1} + \cdots + y_{l+1})$. $\square$

Theorem 3.3. *For any connected graph G we have $h_2^B(G) \geq s_{1/2}(G)$.*

Proof. Let q_i denote the vertex asked in the ith step and P_i denote the set of vertices which can possibly be endpoints of the path before Step i (i.e. $P_1 = V$). Now we show a very simple strategy of the adversary: choose the answer which gives the largest possible P_{i+1}. In case more answers give the same cardinality, then choose arbitrarily with priority given to the answer "the path does not pass through this vertex", except never answer "yes, this is the endpoint", only if that is the only possible answer (in which case this query was redundant anyway).

We claim that when an algorithm finds t, the end of the path against this adversary, the set C of the vertices asked form a 1/2-separator (or have cardinality $|V| - 1$). At first we show that they form a separator. Obviously t is the only element of P_i. If its connected component in the graph spanned by $V \setminus C$ contained another vertex, that would be a possible endpoint too, a contradiction.

The above algorithm answers that the path does not go through q_i except when q_i is a cut-vertex of the graph spanned by P_i. The answer in Step i partitions P_i into $i_m \geq 2$ parts of size $1 = x_{i,1} \leq x_{i,2} \leq \cdots \leq x_{i,m_i}$ (where the first part is a set containing only q_i, the rest of the parts are the components of the graph induced by $P_i \setminus \{q_i\}$). The adversary chooses a biggest of these parts, i.e. one of size x_{im_i} to be the remaining set P_{j+1} and the rest is cut down. Thus sets with the following sizes are cut down in this order (altogether we have l cuts): $x_{11}, x_{1,2}, \ldots, x_{1,m_1-1}, x_{2,1}, x_{2,2}, \ldots, x_{2,m_2-1}, \ldots, x_{l,1}, x_{l,2}, \ldots, x_{l,m_l-1}$ and finally the only remaining part is of size $x_{l,m_l} = 1$ when the algorithm finds the end-vertex. Let us denote $y_i = x_{i,1} + x_{i,2} + \cdots + x_{i,m_i-1}$ and $y_{l+1} = x_{l,m_l} = 1$. By the algorithm we have $x_{i,j} \leq x_{i,m_i} = y_{i+1} + \cdots + y_l + y_{l+1}$ for any i and j. The sum of all the parts $y_1 + y_2 + \cdots + y_l + y_{l+1} = N$ as the corresponding vertex sets partition the vertex set of G.

We apply Lemma 3.2 to the x_i's and y_i's coming from the algorithm and setting $z = \lceil N/2 \rceil$. As all the $x_{i,j}$'s refer to sizes of connected components in the graph spanned by $V \setminus C$ or to size-one sets of asked vertices, the above lemma shows that we can select some components and asked vertices such that the size of their union is at most $\lceil N/2 \rceil$. As already $x_{1,1}$ corresponds to the first asked vertex, and it is always chosen by the lemma, the size of the union of the components is at most $\lceil N/2 \rceil - 1 \leq N/2$. Also, the union of the remaining components has size at most $N - \lceil N/2 \rceil \leq N/2$. Thus C is indeed a 1/2-separator, as claimed. $\square$

Corollary 3.4. *$h_2^B(G_d(n)) \geq n^{d-1}/3d$.*

Theorem 3.1 together with Theorem 3.3 imply that $h_1^B(G^{4,4}) \geq h_2^B(G) \geq s_{1/2}(G)$ for any grid graph G. One can easily see that if 4 divides n and G is the $n/4 \times n/4 \times n \times \cdots \times n$ grid graph, then $G_d(n) = G^{4,4}$. We need to show a lower bound on the size of separators in G.

Claim 3.5. *Any α-separator in G has cardinality at least $(1-\alpha)n^{d-1}/16d$.*

For sake of brevity, we omit the proof (it goes the same as the proof of Theorem 1.7, just the axis chosen at the beginning should be orthogonal to the first two axis).

Corollary 3.6. *$h_1^B(G_d(n)) \geq n^{d-1}/48d$.*

Theorem 1.6 summarizes the above results. The lower and upper bounds are quite close, if we consider d as a fixed number, then the theorem gives exact asymptotics in n for the needed number of queries.

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